# UNIVERSITY COLLEGE LONDON

## EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3801**

ASSESSMENT PATTERN : **MATH3801A**

MODULE NAME : **Logic**

DATE : **07-May-13**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2012/13-MATH3801A-001-EXAM-2


**TURN OVER**

*All questions may be attempted, but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Define the set of formulae, $\mathcal{L}$, in the general first order predicate language.
   (b) Define the weight of a string.
   (c) Prove that, if $\alpha$ is a formula, then no proper initial segment of $\alpha$ is a formula. (You may assume that any formula has weight $-1$).
   (d) Consider the following strings, where $x$, $y$ are variables, $P$ is a unary predicate, and $Q$ is a binary predicate:
   (i) $\neg Py \Rightarrow \forall x Px Qxy$
   (ii) $\forall x \Rightarrow Px \forall y Qxy$
   For each string above, determine whether or not it is a formula, justifying your answer.

2. (a) Give the definition of a valuation on the set of propositions, $\mathcal{L}_0$.
   (b) Define what it means to say that a proposition is a tautology.
   (c) Use the semantic tableaux method to determine whether or not each of the following propositions is a tautology (where $\alpha$, $\beta$, $\gamma$ are assumed to be distinct primitive propositions). If a proposition is not a tautology, describe a valuation for which it fails to be true.
   (i) $(\neg(\alpha \Rightarrow (\neg\beta))) \Rightarrow (\alpha \Rightarrow \beta)$
   (ii) $(\alpha \vee (\beta \wedge \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow \gamma)$
   (iii) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow (\beta \Rightarrow (\alpha \Rightarrow \gamma))$

3. (a) Define what it means to have a proof of a proposition $\alpha$ from a set of propositions $S$, giving also the set of axioms and the rule of deduction.
   (b) State the Deduction Theorem for propositional logic.
   (c) Use the Deduction Theorem to show the following:
   $$\vdash \; (\neg(\alpha \Rightarrow (\neg\beta))) \Rightarrow \beta$$
   (d) (i) State what it means for a set $S$ of propositions to be consistent.
   (ii) Prove that, if $S$ is a consistent set of propositions and $\alpha$ is any proposition, then at least one of $S \cup \{\alpha\}$ or $S \cup \{\neg\alpha\}$ is consistent.

4. (a) Define an $\mathcal{L}(\Pi, \Omega)$-structure, where $\Pi$ is a given set of predicate symbols and $\Omega$ is a given set of functional symbols, with assigned arities.
   (b) Describe a theory in a suitably defined first order predicate language $\mathcal{L}(\Pi, \Omega)$, such that a structure $U$ is a (normal) model of the theory if and only if $U$ is a graph.
   (c) In each case below, by extending, where necessary, the sets of predicate symbols, functional symbols and sentences used in the theory described in part(b), write down a theory that has as (normal) models precisely:
   (i) all graphs in which no pair of vertices is connected by an edge.
   (ii) all graphs consisting of at most 3 vertices.
   (d) State and prove the Upper Löwenheim-Skolem Theorem for first order predicate logic.
   (For $S$ a set of sentences in a first order predicate language, you may assume that, if every finite subset of $S$ has a model, then so does $S$.)

5. (a) Define the notion of a register machine, giving also a description of what a program is and of the types of instructions associated to the states of a program.
   (b) State what it means for a function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ to be computable.
   (c) Show that the following functions are computable:
   (i) $f : \mathbb{N}_0 \to \mathbb{N}_0\,, \quad f(m) = m + 5$
   (ii) $f : \mathbb{N}_0 \to \mathbb{N}_0\,, \quad f(m) = 5$
   (iii) $f : \mathbb{N}_0^2 \to \mathbb{N}_0\,, \quad f(m, n) = 5n + 1$